Lecture 16-18

Definition 1. *An n-vertex graph with average degree d and girth g is called* ***Moore Graph*** *if $n = n_0(d, g)$.*

Theorem 1 (Hoffman-Singleton)**.** *If there exists a d-regular Moore graph of girth $g = 5$, then $d \in \{2, 3, 7, 57\}$.*

Proof. Let G be such a graph. Then $|V(G)| = n = n_0(d, 5) = 1+d+d(d-1) = d^2+1$. Since G is Moore graph with girth 5, the diameter of d is 2. Thus u, v have a unique common neighbor for all non-adjacent u, v. Let A be the adjacency matrix of G. Then $A^2 + A - (d-1)I = J$. Here J is $n \times n$ matrix with all entries equal 1.
Note that $A\mathbb{1} = d\mathbb{1}$, since A is d-regular. It's easy the multiplicity of eigenvalues d is 1. Suppose v is an eigenvector of A with respect to some eigenvalue $x \neq d$. Then $v^T\mathbb{1} = 0$. Also note that $J\mathbb{1} = n\mathbb{1}$ and $rank(J) = 1$. So J has eigenvalues $0^{(n-1)}$ and $n^{(1)}$, and thus $Jv = 0$.
Then $A^2v + Av - (d-1)v = Jv$, which implies $x^2 + x - (d-1) = 0$. This equation has two roots $x_{1,2} = \frac{1}{2}(-1 \pm \sqrt{4d-3})$. So we can suppose A has eigenvalues $d^{(1)}, x_1^{(r)}, x_2^{(s)}$. Then $r + s = n - 1 = d^2$.
Note that $0 = tr(A) = d + rx_1 + sx_2 = d - \frac{r+s}{2} + \frac{r-s}{2}\sqrt{4d-3}$.
If x_1, x_2 are irrational, then $r = s, d = r$ and so $2r = d^2 = r^2$, which yields $r = s = d = 2$.
If x_1, x_2 are rational, then $\sqrt{4d-3} = m$ must be an integer and so r is also an integer. Then $d = \frac{m^2+3}{4}$ and $s = d^2 - r = (\frac{m^2+3}{4})^2 - r$.
Thus $0 = d - \frac{r+s}{2} + \frac{r-s}{2}\sqrt{4d-3} = \frac{m^2+3}{4} - (\frac{m^2+3}{4})^2/2 + (2r - (\frac{m^2+3}{4})^2)m/2$. That is,

$$m^5 + m^4 + 6m^3 - 2m^2 + (9 - 64r)m = 15.$$

In particular, m is a factor of 15. Also note that $d = \frac{m^2+3}{4} \geqslant 2$. So $m \in \{3, 5, 15\}$. And so $d \in \{3, 7, 57\}$. $\square$

Remark. It's known that there exists d-regular Moore graphs with girth 5 for $d = 2, 3, 7$. But this is open for $d = 57$.

Definition 2. *The* ***k-core*** *of a graph G is the largest subgraph of min-degree at least k. The k-core can be found by repalcing deleting vertices of degree less than k. So the k-core may be empty.*

Fact 1. *k-core is well-defined.*

Proof. Exercise. □

Let $H_{k,n}$ be the n-vertex complement of K_{n-k}. Clearly, $H_{k,n}$ has empty $(k+1)$-core and $e(H_{k,n}) = k(n-k) + \binom{k}{2}$.

Lemma 2. *If a graph G is an n-vertex grapg with at least $k(n-k) + \binom{k}{2}$ edges, then G has a non-empty $(k+1)$-core unless $G = H_{k,n}$.*

Lemma 3. *Let H be a graph comprising a cycle C with a chord, and let (A, B) be a non-trivial pritition of $V(H)$. Then for $\ell < |V(H)|$, there is a graph P of length ℓ with an endpoint in A and the other endpoint in B. Unless H is bipartite and (A, B) is a bipartition of H and ℓ even.*

Proof. Assume $C = 0 - 1 - 2 - \cdots - (n-1) - 0$ with chord $0 - r$.
Let m be smallest integer k such that there is no $A - B$ path of length k only using edges in $E(C)$.
If we let χ be the characteristic function of set A, then $\chi(j) = \chi(j+m)$ for all j. And so $m|n$.
Note that by the definition of m, for all ℓ that $\ell \nmid j$, there exists $A - B$ path of length ℓ. So we only need to consider $A - B$ paths of length multiples of m.
Case 1. The chord $0 - r$ satisfies $1 < r \leqslant m$.
Since $m \nmid m + r - 1$, there is some $-m < j \leqslant 0$ such that $\chi(j) \neq \chi(j + m + r - 1) = \chi(j + km + r - 1)$ for all integer k.
Consider the path $P = j, j+1, \cdots, 0, r, r+1, \cdots, j+m+r-1, \cdots, j+km+r-1$, where $1 \leqslant k \leqslant \frac{n}{m} - 1$. It's an $A - B$ path of length km.
Case 2. $m < r < n - m$.
Given j that $-m \leqslant j \leqslant 0$, consider 2 paths $P = j, j+1, \cdots, 0, r, r-1, \cdots, r, r-1, \cdots, r-j-m+1$ and $Q = m+j, m+j-1, \cdots, 0, r, r+1, r-j-1$. If P or Q is

an $A-B$ path (of length m), then we can extend it to an $A-B$ path of length km, by m vertices at a time, until $km+1 \geqslant n-2(m-1)$. Thus we can find $A-B$ path of length km for every $1 \leqslant k \leqslant \frac{n}{m}-1$.
Now we consider $\chi(j) = \chi(r-j+m+1)$ and $\chi(m+j) = \chi(r-j-1)$ for all $-m \leqslant j \leqslant 0$. Then $\chi(v+2) = \chi(v+2)$ where $v = r-j-1 \in [r-1, r+m-1]$ as $-m \leqslant j \leqslant 0$, which implies $\chi(v) = \chi(v+2)$ for all v. So $m=2$ and $2 \mid n$, which means the vertices of C are alternatively in A and in B (So all the paths on C of odd lengths are $A-B$ paths). In this case, if the chord $0-r$ are in the same part, then one can easily check that H has all $A-B$ paths. Otherwise, the chord $0-r$ join A and B then (A,B) is the bipartition of H. □

Theorem 4 (Bondy-Simonovits)**.** *For all* $n \geqslant (3(k-1))^k$, $\mathrm{ex}(n, C_{2k}) \leqslant 2kn^{1+\frac{1}{k}}$.

Proof. Suppose G is an n-vertex C_{2k}-free graph with $e(G) > 2kn^{1+\frac{1}{k}}$. Then G has a bipartite subgraph H' with $e(H') > kn^{1+\frac{1}{k}}$. Furthermore, H' has a bipartite subgraph H with $\delta(H) > kn^{\frac{1}{k}}$. Let T be a BFS-tree of H with root x and let $L_i = \{y \in V(H) : d_H(x,y) = i\}$ for $i = 0, 1, \cdots, k$.
Claim 1. $e_H(L_{i-1}, L_i) \leqslant (k-1)(|L_{i-1}| + |L_i|)$ for $i = 1, \cdots, k$.
Proof of Claim 1. Clearly, it's for $i=1$.
Suppose $e_H(L_{i-1}, L_i) > (k-1)(|L_{i-1}|+|L_i|)$ for some $i = 2, \cdots, k$. Then $H(L_{i-1}, L_i)$ has a bipartite subgraph H_1 with $\delta(H_1) \geqslant k$. And then H_1 contains an even cycle C (of length at least $2k$) with a chord.
Let $A = V_{i-1} \cap V(C)$ and $B = V_i \cap V(C)$, then (A,B) is a bipartition of C. Let y be the vertex that is farthest from root x such that every vertex of Y is a T-descendant of y. The paths inside T that connect y to A branch at y. Pick one such branch, defined by some child z of y, and let A' be the set of the T-descendants of z that lie in A. Let $B' = V(C) - A'$. Since $A - A' \neq \emptyset$, B is not an independent set of C.
Let ℓ be the distance between x and y. We have $\ell < i$ and $2k-2i+2\ell < 2k \leqslant |V(C)|$. By Lemma 3, we can find a path $P \subset C$ of length $2k-2i+2\ell$ that starts in some $a \in A'$ and $b \in B'$. Since the length of P is even, we have $b \in A$. Let P_a and P_b be the unique paths in T that connect y to respectly a and b. They intersect only in the vertex y by the definition of A'. Thus $P \cup P_a \cup P_b$ forms a C_{2k} in H, a contradiction. □
Claim 2. $|L_i| \geqslant n^{\frac{1}{k}}|L_{i-1}|$, for $i = 1, \cdots, k$.
Proof of Claim 2. We prove it by induction on i.
Base case $i=1$ is trivial.

Suppose $i \geqslant 2$ and claim holds for all $j < i$, then by Claim 1,

$$\begin{aligned} kn^{\frac{1}{k}}|L_{i-1}| &\leqslant \sum_{v \in L_{i-1}} d_H(v) = e_H(L_{i-2}, L_{i-1}) + e_H(L_{i-1}, L_i) \\ &\leqslant (k-1)(|L_{i-2}| + 2|L_{i-1}| + |L_i|) \\ &\leqslant (k-1)(3|L_{i-1}| + |L_i|). \end{aligned}$$

Thus $|L_i| \geqslant [\frac{k}{k-1}n^{\frac{1}{k}} - 3]|L_{i-1}| \geqslant n^{\frac{1}{k}}|L_{i-1}|$. □

So by Claim 2, we have $L_k \geqslant n$, a contradiction! □

Conjecture 5 (Erdös-Simonovits)**.** *$\mathrm{ex}(n, C_{2k}) = \Theta(n^{1+\frac{1}{k}})$.*

The best upper bound was recently obtained.

Theorem 6 (Bukh-Jiang, 2016)**.** *$\mathrm{ex}(n, C_{2k}) \leqslant 80\sqrt{k \log k}\, n^{1+\frac{1}{k}} + 10k^2 n$ for all large n.*

Theorem 7. *Let k be an integer and let G be an n-vertex graph. If $e(G) = \Omega(kn)$, then G contains k even cycles of consecutive lengths, say $C_{2r}, C_{2r+2}, \cdots, C_{2r+2k-2}$ for some r.*

Proof. Exercise. □

Theorem 8 (Liu-Ma)**.** *Every graph G with $\delta(G) \geqslant 2k+1$ has k even cycles of consecutive lengths.*

Remark. $2k+1$ is tight, by considering $G = K_{2k+2}$.

Lemma 9 (Posá's Lemma)**.** *Let G be a graph satisfying that $|N(X)| > 2|X|$ for every $X \subset V(G)$ with $|X| \leqslant t$. Then G contains a cycles of length at least $\min\{3t, n\}$ with a chord.*

Proof. Exercise. □